

AMENABILITY OF BANACH ALGEBRAS OF COMPACT OPERATORS

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ABSTRACT

In this paper we study conditions on a Banach space X that ensure that the Banach algebra $\mathcal{K}(X)$ of compact operators is amenable. We give a symmetrized approximation property of X which is proved to be such a condition. This property is satisfied by a wide range of Banach spaces including all the classical spaces. We then investigate which constructions of new Banach spaces from old ones preserve the property of carrying amenable algebras of compact operators. Roughly speaking, dual spaces, predual spaces and certain tensor products do inherit this property and direct sums do not. For direct sums this question is closely related to factorization of linear operators. In the final section we discuss some open questions, in particular, the converse problem of what properties of X are implied by the amenability of $\mathcal{K}(X)$.

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0. Introduction

Amenability is a cohomological property of Banach algebras which was introduced in [J]. The definition is given below. It may be thought of as being, in some ways, a weak finiteness condition. For example, amenability of C^* -algebras is equivalent to nuclearity, see [Haa]. Also, a group algebra, $L^1(G)$, is amenable if and only if the locally compact group, G , is amenable, see [J], and many theorems valid for finite or compact groups have weaker generalizations to amenable groups but to no larger class. This equivalence is the origin of the term for Banach algebras. However, in some situations amenability is not a finiteness condition. For example, a uniform algebra is amenable if and only if it is self-adjoint, see [Sh], and, for finite dimensional Banach algebras, amenability is equivalent to semisimplicity, [Li, Theorem 1.3].

The significance of amenability for some classes of Banach algebras suggests the question as to what it means for other Banach algebras. In this paper we investigate the amenability of the algebras of compact and of approximable operators on the Banach space X . This was begun in [J], where it is shown that $\mathcal{K}(X)$ is amenable if X is ℓ_p , $1 < p < \infty$, or $C[0, 1]$. ($\mathcal{K}(X)$ denotes the algebra of compact operators on X and $\mathcal{F}(X)$ the algebra of approximable operators.) Relevant properties of Banach spaces, such as the approximation property, are now understood better than they were when [J] was written and so we are able to make more progress.

We have not yet found such clear characterizations of amenability for the algebras of approximable and compact operators as are known for classes of algebras mentioned in the first paragraph. It does appear though that amenability of $\mathcal{F}(X)$ and $\mathcal{K}(X)$ may be equivalent to certain approximation properties for X . One immediate observation is that, since amenable Banach algebras have bounded approximate identities, if the algebra of compact operators on X is amenable, then, by [D, Theorem 2.6], X has the bounded compact approximation property and, if the algebra of approximable operators is amenable, then X has the bounded approximation property. Moreover, results in [G&W] and [Sa] show that, if $\mathcal{K}(X)$ is amenable, then X^* has the bounded compact approximation property and, if $\mathcal{F}(X)$ is amenable, then X^* has the bounded approximation property. It follows that, if $\mathcal{F}(X)$ is amenable, then $\mathcal{K}(X) = \mathcal{F}(X)$.

Amenability of $\mathcal{F}(X)$ is not equivalent to X or X^* having the bounded approximation property however, as examples in the paper show. Some sort of

symmetry also seems to be required. In Section 3 we formulate a symmetrized approximation property, called property (A), such that, if X has property (A), then $\mathcal{F}(X)$ is amenable. This formulation is an abstract version of the argument used in [J]. We show that, if X has a shrinking, subsymmetric basis, then it has property (A) and hence $\mathcal{F}(X)$ is amenable. Many spaces which do not have such a basis also have property (A).

The necessity of some sort of symmetry becomes apparent when we consider the stability of the class of spaces X such that $\mathcal{F}(X)$ is amenable. Subject to some restrictions, this class of spaces is closed under tensor products and taking duals, as is shown in Sections 2 and 5. However, it is not closed under direct sums or passing to complemented subspaces, see Section 6. The results in Sections 5 and 6 depend on some new stability properties for amenable Banach algebras which we establish in those sections.

Many questions remain to be answered before we understand fully the connection, if any, between amenability of $\mathcal{F}(X)$ or $\mathcal{K}(X)$ and approximation properties of X . These questions are discussed in the last section of the paper. We do not investigate other homological properties of $\mathcal{F}(X)$ and $\mathcal{K}(X)$. One other such property has been studied in [Ly].

We now give the definition of amenability for Banach algebras. It is made in terms of Banach modules and derivations. Recall that, for a Banach algebra \mathcal{A} , a Banach space X is a Banach \mathcal{A} -bimodule if X is a \mathcal{A} -bimodule and there is a constant K such that $\|a.x\| \leq K\|a\|\|x\|$ and $\|x.a\| \leq K\|a\|\|x\|$ for each a in \mathcal{A} and x in X . If X is a Banach \mathcal{A} -bimodule, then the dual space, X^* , is a Banach \mathcal{A} -bimodule with the actions defined by $\langle a.x^*, x \rangle = \langle x^*, x.a \rangle$ and $\langle x^*.a, x \rangle = \langle x^*, a.x \rangle$, for a in \mathcal{A} , x in X and x^* in X^* . A derivation into an \mathcal{A} -bimodule X is a linear map $D: \mathcal{A} \rightarrow X$ such that $D(ab) = a.D(b) + D(a).b$, for all a, b in \mathcal{A} . If x belongs to X , then the map $a \mapsto a.x - x.a$ is a derivation into X . Such derivations are called inner.

Definition 0.1: The Banach algebra \mathcal{A} is amenable if, for every Banach \mathcal{A} -bimodule X , every continuous derivation $D: \mathcal{A} \rightarrow X^*$ is inner.

See [J, Section 5], or [B&D, Definition VI.2].

This definition will sometimes be used directly but we will often use another characterization of amenability, namely that \mathcal{A} is an amenable Banach algebra if and only if $\mathcal{A} \widehat{\otimes} \mathcal{A}$ has an approximate diagonal. An approximate diagonal is a

bounded net, $\{d_\lambda\}_{\lambda \in \Lambda}$, in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that

$$\lim_{\lambda \rightarrow \infty} \|a \cdot d_\lambda - d_\lambda \cdot a\| = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \|\pi(d_\lambda)a - a\| = 0 \quad (a \in \mathcal{A}),$$

where π denotes the product map $\mathcal{A} \widehat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ and module actions on $\mathcal{A} \widehat{\otimes} \mathcal{A}$ are defined by $a \cdot (b \otimes c) = (ab) \otimes c$ and $(b \otimes c) \cdot a = b \otimes (ca)$, for a, b and c in \mathcal{A} . If we define a product on $\mathcal{A} \widehat{\otimes} \mathcal{A}$ by $(a \otimes b)(c \otimes d) = ac \otimes db$, then the first of these conditions can also be stated as

$$\lim_{\lambda \rightarrow \infty} \|(a \otimes 1 - 1 \otimes a)d_\lambda\| = 0 \quad (a \in \mathcal{A}),$$

where 1 is a formally adjoined unit. Approximate diagonals are useful, for example, when we show that, if X has property (A), then $\mathcal{F}(X)$ is amenable.

1. Notation

We begin by establishing notation. Throughout, X and Y will denote (infinite dimensional) Banach spaces and X^* the space of bounded linear functionals on X with its usual norm. Small letters x etc. will denote elements in X , whereas x^* etc. will denote elements in X^* . We will consider the following classes of operators:

$$\begin{aligned} F(X, Y) &= \{\text{finite rank operators } X \rightarrow Y\}, \\ \mathcal{N}(X, Y) &= \{\text{nuclear operators } X \rightarrow Y\}, \\ \mathcal{F}(X, Y) &= \text{uniform closure of } F(X, Y) \\ &= \{\text{approximable operators } X \rightarrow Y\}, \\ \mathcal{K}(X, Y) &= \{\text{compact operators } X \rightarrow Y\}, \\ \mathcal{I}(X, Y) &= \{\text{integral operators } X \rightarrow Y\}, \\ \mathcal{B}(X, Y) &= \{\text{bounded operators } X \rightarrow Y\}. \end{aligned}$$

We shall write $F(X)$ instead of $F(X, X)$ etc.

These are all two-sided operator ideals in $\mathcal{B}(X, Y)$, and when $X = Y$ they are, except $F(X)$, Banach algebras in their natural norms. We refer the reader to any of [D&U], [Pie], [Pis] for details.

Finite rank operators will, when convenient, be written as tensors, that is, if $x_1^*, \dots, x_n^* \in X^*$ and $y_1, \dots, y_n \in Y$, we shall denote the operator $x \rightarrow \sum x_i^*(x)y_i$ by $\sum y_i \otimes x_i^*$.

If $S \in \mathcal{B}(X, Y)$ we denote the adjoint map in $\mathcal{B}(Y^*, X^*)$ by S^α , i.e.

$$\langle S(x), y^* \rangle = \langle x, S^\alpha(y^*) \rangle \quad (x \in X, y^* \in Y^*).$$

If $M \subseteq \mathcal{B}(X, Y)$ we define $M^\alpha \subseteq \mathcal{B}(Y^*, X^*)$ by

$$M^\alpha = \{T^\alpha \mid T \in M\}.$$

This should not be confused with the notation for dual space. For instance, if X has Grothendieck's approximation property, then $\mathcal{N}(X)^* = \mathcal{B}(X^*)$, whereas $\mathcal{N}(X)^\alpha$ is the set of so-called X -nuclear operators on X^* .

We shall use the concepts left approximate identity, bounded left approximate identity etc. in accordance with [B&D].

2. Tensor products

It is of course important to be able to form new Banach spaces from old ones while preserving the property of carrying amenable algebras of compact operators. The first case to be considered is that of taking tensor products because many important spaces can be viewed as appropriate tensor products, for instance L_p -spaces with values in a Banach space. We shall here investigate whether amenability of $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ implies amenability of $\mathcal{K}(Z)$, when Z is the completion of $X \otimes Y$ in some crossnorm topology. An obvious approach to this problem is to try to show that $\mathcal{K}(X) \otimes \mathcal{K}(Y)$ is a dense subalgebra of $\mathcal{K}(Z)$ and then to deduce amenability of $\mathcal{K}(Z)$ from that of $\mathcal{K}(X) \widehat{\otimes} \mathcal{K}(Y)$ by an appeal to [J, Corollary 5.5]. This program is considerably easier to carry through if X and Y have the approximation property. However, rather than making this assumption, we prefer to work with approximable operators instead of compact operators. The definition to follow describes what is needed for above mentioned program to work.

Definition 2.1: Let X and Y be Banach spaces and let α be a crossnorm on $X \otimes Y$. Denote the completion by $X \otimes_\alpha Y$. We call $X \otimes_\alpha Y$ a **tight** tensor product of X and Y , if the following two conditions hold:

- (i) There is $K > 0$ so that for all $S \in \mathcal{F}(X)$, $T \in \mathcal{F}(Y)$ the operator on $X \otimes Y$ given by

$$(S \otimes T)x \otimes y = Sx \otimes Ty \quad (x \in X, y \in Y)$$

has α - operator norm not exceeding $K\|S\|\|T\|$.

- (ii) $\text{span}\{S \otimes T \mid S \in \mathcal{F}(X), T \in \mathcal{F}(Y)\}$ is dense in $\mathcal{F}(X \otimes_\alpha Y)$.

Remark: The condition (i) is apparently weaker than Grothendieck's \otimes -norm condition [Gr, Ch.1.3] in that it only concerns finite rank operators on a tensor product into itself.

With this definition we have the obvious:

THEOREM 2.2: *Suppose that $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are amenable and that $X \otimes_\alpha Y$ is a tight tensor product. Then $\mathcal{F}(X \otimes_\alpha Y)$ is amenable.*

Proof: [J, Corollary 5.5]. ■

To apply this theorem we need to be able to recognize tight tensor products. The following easy proposition is helpful. It shows that, as usual when dealing with tensor products, it is important to be able to identify $(X \otimes_\alpha Y)^*$. We shall view $(X \otimes_\alpha Y)^*$ as a subspace of $\mathcal{B}(Y, X^*)$ (or equivalently of $\mathcal{B}(X, Y^*)$). We give $\mathcal{B}(Y, X^*)$ the canonical structure as a right Banach module over $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, that is, the module actions are the restrictions of the canonical actions of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$.

PROPOSITION 2.3: *Let X and Y be Banach spaces and let α be a crossnorm on $X \otimes Y$. Then $X \otimes_\alpha Y$ is a tight tensor product if and only if the following two conditions hold:*

- (i) $(X \otimes_\alpha Y)^*$ is a right Banach $\mathcal{F}(X)$ - and $\mathcal{F}(Y)$ -submodule of $\mathcal{B}(Y, X^*)$
- (ii) $X^* \otimes Y^*$ is norm dense in $(X \otimes_\alpha Y)^*$.

Proof: (i) Let $z = \sum x_i \otimes y_i \in X \otimes Y$, let $\Phi \in (X \otimes_\alpha Y)^*$, and let $S \in \mathcal{F}(X)$, $T \in \mathcal{F}(Y)$. Then

$$\begin{aligned} \langle S \otimes T z, \Phi \rangle &= \sum \langle Sx_i \otimes Ty_i, \Phi \rangle \\ &= \sum \langle Sx_i, \Phi Ty_i \rangle \\ &= \langle z, S^a \Phi T \rangle, \end{aligned}$$

so that $S \otimes T$ is α -bounded with $\|S \otimes T\|_\alpha \leq K \|S\| \|T\|$ if and only if the Banach module properties hold with module constants $K_X K_Y \leq K$.

(ii) We shall use the identification $\mathcal{F}(Z) = Z \overset{\vee}{\otimes} Z^*$. The canonical map $\mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes_\alpha Y)$ then becomes

$$(x \overset{\vee}{\otimes} x^*) \otimes (y \overset{\vee}{\otimes} y^*) \rightarrow (x \otimes_\alpha y) \overset{\vee}{\otimes} (x^* \otimes y^*).$$

Using the injective property of $\overset{\vee}{\otimes}$, it is now clear that the image of $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ is dense if and only if $X^* \otimes Y^*$ is dense in $(X \otimes_\alpha Y)^*$. ■

Recall that a crossnorm is called **reasonable** if the dual norm is also a crossnorm. In this case tightness is particularly easy to describe.

COROLLARY 2.4: *Suppose that α is a reasonable crossnorm on $X \otimes Y$ and that the module property 2.3(i) holds. Then $X \otimes_\alpha Y$ is tight if and only if*

$$(X \otimes_\alpha Y)^* = X^* \otimes_{\alpha^*} Y^*,$$

where α^* denotes the dual norm.

With Proposition 2.3 at hand we can now give conditions for tightness for some important tensor products. From 2.3(ii) it is not surprising that the Radon–Nikodym property enters the picture.

THEOREM 2.5: *Let X and Y be Banach spaces, let $[0, 1]$ be the unit interval, and let (Ω, Σ, μ) be a σ -finite measure space. Then*

(W) *The following are equivalent:*

(i) $X \overset{\vee}{\otimes} Y$ *is tight for all* X .

(ii) $C([0, 1], Y)$ *is a tight tensor product of* $C[0, 1]$ *and* Y .

(iii) Y^* *has RNP.*

(P) $X \widehat{\otimes} Y$ *is tight if and only if* $\mathcal{F}(Y, X^*) = \mathcal{B}(Y, X^*)$.

(M) $L_p(\mu, X)$, $1 \leq p < \infty$ *is a tight tensor product of* $L_p(\mu)$ *and* X *if and only if* X^* *has RNP with respect to* μ .

Proof: The identification of $(X \otimes_\alpha Y)^*$ with a subspace of $\mathcal{B}(Y, X^*)$ gives in the cases (W) and (P) $\mathcal{I}(Y, X^*)$ and $\mathcal{B}(Y, X^*)$ respectively, so the module property 2.3(i) is obvious for these tensor products. Next, let $S \in \mathcal{F}(L_p(\mu))$ and $T \in \mathcal{F}(X)$. From the proof of Proposition 2.3(i) it follows that it is enough to show the submultiplicativity of the module norm for Φ belonging to a norm determining subset of $L_p(\mu, X)^*$. Let $1/p + 1/p' = 1$. Since $L_{p'}(\mu, X^*)$ is isometrically embedded in $L_p(\mu, X)^*$ and since $L_p(\mu, X)$ is isometrically embedded in $L_{p'}(\mu, X^*)^*$ it suffices to look at $\Phi \in \mathcal{B}(L_p(\mu), X^*)$ coming from an element $\mathbf{g} \in L_{p'}(\mu, X^*)$. With the identifications being made we have

$$\Phi(f) = \int_{\Omega} f \mathbf{g} \, d\mu \quad (f \in L_p(\mu)).$$

Then for $S \in \mathcal{B}(L_p(\mu))$ and $T \in \mathcal{B}(X)$

$$T^\alpha \Phi S(f) = \int_{\Omega} S(f) T^\alpha \circ \mathbf{g} \, d\mu.$$

An appeal to the vector valued version of Hölders inequality, gives the desired norm inequality.

We now consider the statement 2.3(ii) in our three cases. First we look at (W). Since $(X \overset{\vee}{\otimes} Y)^* = \mathcal{I}(Y, X^*)$ we are asking whether the finite rank operators $X^* \otimes Y^*$ are dense in $\mathcal{I}(Y, X^*)$ in the integral norm. The implication (i) \Rightarrow (ii) is obvious and (iii) \Rightarrow (i) is valid because, if Y^* has RNP, then $\mathcal{I}(Y, X^*) = \mathcal{N}(Y, X^*)$ isometrically, [D&U, Theorem VI.4.8, Corollary VIII.2.10]. The implication (ii) \Rightarrow (iii) is true because, under the assumption (ii), $F(C[0, 1], Y^*)$ is dense in $\mathcal{I}(C[0, 1], Y^*)$. (We are here using the symmetric rôles of $C[0, 1]$ and Y .) By [D&U, Theorem VI.3.12, Corollary VIII.2.10] every absolutely summing operator $C[0, 1] \rightarrow Y^*$ is nuclear since, by [Pie, Proposition 10.3.1], $\mathcal{I}(C[0, 1], Y^*) = \mathcal{N}(C[0, 1], Y^*)$. The RNP of Y^* is now the content of [D&U, Corollary VI.4.6].

In the case (P) we just have to observe that $(X \widehat{\otimes} Y)^* = \mathcal{B}(Y, X^*)$ and $\text{cl}(X^* \otimes Y^*) = \mathcal{F}(Y, X^*)$.

Finally, as already noticed, $L_{p'}(\mu, X^*)$ is isometrically isomorphic to a subspace of $L_p(\mu, X)^*$. As a consequence $L_p(\mu, X)$ is tight if and only if $L_p(\mu, X)^* = L_{p'}(\mu, X^*)$. But this is equivalent to X^* having RNP with respect to μ , see for instance [D&U, Theorem IV.1.1].

From a classical theorem by Pitt [Pit] we get an immediate consequence of Theorem 2.5.(P).

COROLLARY 2.6: $\ell_p \widehat{\otimes} \ell_q$ is tight if and only if $1/p + 1/q < 1$.

COROLLARY 2.7: If X^* has RNP and $\mathcal{F}(X^*)$ is amenable, then $\mathcal{F}(\mathcal{F}(X))$ is amenable.

Proof: By Corollary 5.3 below, amenability of $\mathcal{F}(X^*)$ forces amenability of $\mathcal{F}(X)$. The identification $\mathcal{F}(X) = X \overset{\vee}{\otimes} X^*$ shows that $\mathcal{F}(X)$ is a tight tensor product of X and X^* . ■

3. Diagonals for $M_n(\mathbb{C})$

For many Banach spaces X , in particular the classical spaces, it is possible to prove that $\mathcal{K}(X)$ is amenable as a consequence of a uniform local structure of X , that is, as a consequence of a property of finite dimensional subspaces. Before we set the scenario in which this approach will work we shall take a closer look at finite dimensional spaces. It is well known and easy to prove that $M_n(\mathbb{C})$ is amenable. In this section we shall view this in terms of faithful irreducible

representations of finite groups. However, rather than speaking about faithful representations we shall consider finite subgroups of $Gl_n(\mathbb{C})$. Likewise, we shall express irreducibility as a property of the embedding of the group into $M_n(\mathbb{C})$.

LEMMA 3.1: *Let $\mathcal{D}: \mathcal{G} \rightarrow Gl_n(\mathbb{C})$ be an n -dimensional representation of a group \mathcal{G} . Then \mathcal{D} is irreducible if and only if $\text{span } \mathcal{D}(\mathcal{G}) = M_n(\mathbb{C})$.*

Proof: We extend the representation to the group algebra $\mathbb{C}\mathcal{G}$. Then the lemma is an easy consequence of Jacobson's density theorem, [B&D, Theorem 24.10].

■

Henceforth we shall deal with finite subgroups of $Gl_n(\mathbb{C})$ spanning the whole of $M_n(\mathbb{C})$. These we shall call **irreducible $(n \times n)$ -matrix groups**. The connection of such with amenability of $M_n(\mathbb{C})$ is described in the following proposition. The symbols e_{ij} denote as usual the matrix units.

PROPOSITION 3.2: *Let \mathcal{G} be a finite irreducible $(n \times n)$ -matrix group. Then $\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g \otimes g^{-1}$ is equal to the canonical diagonal $d_0 = \frac{1}{n} \sum_{i,j=1}^n e_{ij} \otimes e_{ji}$ for $M_n(\mathbb{C})$. The canonical diagonal d_0 is the only element of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ which is simultaneously a diagonal for $M_n(\mathbb{C})$ and for the opposite algebra $M_n(\mathbb{C})^{op}$.*

Proof: Let $d = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g \otimes g^{-1}$. That d is a diagonal for $M_n(\mathbb{C})$ means

$$(3.1) \quad \sum_{g \in \mathcal{G}} ag \otimes g^{-1} = \sum_{g \in \mathcal{G}} g \otimes g^{-1}a \quad (a \in M_n(\mathbb{C}))$$

and

$$(3.2) \quad \pi(d) = I.$$

Likewise, d being a diagonal for $M_n(\mathbb{C})^{op}$ means

$$(3.3) \quad \sum_{g \in \mathcal{G}} ga \otimes g^{-1} = \sum_{g \in \mathcal{G}} g \otimes ag^{-1} \quad (a \in M_n(\mathbb{C}))$$

and

$$(3.4) \quad \pi_{op}(d) = I,$$

where π_{op} is the opposite multiplication $\pi_{op}(a \otimes b) = ba$.

Since $\text{span } \mathcal{G} = M_n(\mathbb{C})$ it is enough to consider $a \in \mathcal{G}$ and then exploit linearity. We prove (3.3):

$$\begin{aligned} \sum_{g \in \mathcal{G}} ga \otimes g^{-1} &= \sum_{g \in \mathcal{G}} ga \otimes a(ga)^{-1} \\ &= \sum_{u \in \mathcal{G}a} u \otimes au^{-1}. \end{aligned}$$

Since $\mathcal{G}a = \mathcal{G}$, (3.3) follows. The identity (3.1) is proved similarly, and (3.2) and (3.4) are obvious.

Simple computations with matrix units show that d_0 satisfies all of (3.1), ..., (3.4). Now let $d = \sum_i a_i \otimes b_i$ be any element satisfying (3.2) and (3.3) and write $d_0 = \sum_j a'_j \otimes b'_j$. Then

$$\begin{aligned} d &= \sum_i a_i \otimes b_i = \sum_{i,j} a_i \otimes b'_j a'_j b_i \\ &= \sum_{i,j} a_i a'_j \otimes b'_j b_i \\ &= \sum_{i,j} a_i b_i a'_j \otimes b'_j \\ &= \sum_j a'_j \otimes b'_j = d_0, \end{aligned}$$

finishing the proof. Note that (3.1) and (3.4) follows automatically from (3.2) and (3.3), since d_0 satisfies (3.1) and (3.4). ■

(The use of the average $\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g \otimes g^{-1}$ probably dates back to the early days of representation theory. It is a refinement of this which gives the equivalence of amenability of group algebras and the existence of invariant means, [J, Theorem 2.5].)

Example 3.3: We shall several times in the sequel use irreducible matrix groups of the following kind. Let for $t \in \{\pm 1\}^n$ the corresponding $n \times n$ diagonal matrix be denoted by $D(t)$. Let \mathcal{H} be a group of $(n \times n)$ permutation matrices corresponding to a transitive subgroup of the symmetric group S_n . Then

$$\mathcal{G} = \{D(t)\sigma \mid t \in \{\pm 1\}^n, \sigma \in \mathcal{H}\}$$

is an irreducible $(n \times n)$ -matrix group. If $\mathcal{H} = S_n$, then \mathcal{G} is called **the monomial group of degree n** .

4. Amenability as a consequence of an approximation property

In this section we shall develop a method to lift uniformly the diagonals of a matrix algebra to form an approximate diagonal for $\mathcal{F}(X)$. The idea is illustrated by the example $X = L_p(\mu)$. Locally $L_p(\mu)$ looks like ℓ_p^n so we have ‘local’ diagonals. Furthermore, these diagonals are uniformly bounded (by 1). Using a direct limit argument we can form an approximate diagonal for all of $\mathcal{F}(L_p(\mu))$.

This approach will work for all the classical spaces. The definition below is customised to make it work in a rather general situation. To formulate it let us first look at a finite biorthogonal system $\{(x_i, x_j^*) \mid x_i \in X; x_j^* \in X^*; i, j = 1, \dots, n\}$. Using this system we may define a map $E: M_n(\mathbb{C}) \rightarrow \mathcal{F}(X)$ by

$$E((a_{ij})) = \sum_{i,j} a_{ij} x_i \otimes x_j^*.$$

By biorthogonality, E is an algebra homomorphism.

Definition 4.1: Let X be a Banach space. We say that X has property **(A)** if there is a net of finite biorthogonal systems

$$\{(x_{i,\lambda}, x_{j,\lambda}^*) \mid x_{i,\lambda} \in X; x_{j,\lambda}^* \in X^*; i, j = 1, \dots, n_\lambda\} \quad (\lambda \in \Lambda)$$

and corresponding maps

$$E_\lambda: M_{n_\lambda}(\mathbb{C}) \rightarrow \mathcal{F}(X) \quad (\lambda \in \Lambda)$$

such that with $P_\lambda = E_\lambda(I_{n_\lambda})$ the following hold:

- A(i)** $P_\lambda \rightarrow 1_X$ strongly.
- A(ii)** $P_\lambda^\alpha \rightarrow 1_{X^*}$ strongly.
- A(iii)** For each λ there is an irreducible $(n_\lambda \times n_\lambda)$ -matrix group \mathcal{G}_λ such that

$$\sup\{\|E_\lambda(g)\|_{\text{op}} \mid g \in \mathcal{G}_\lambda, \lambda \in \Lambda\} < \infty.$$

We now show how to lift the diagonals of the matrix algebras to $\mathcal{F}(X)$.

THEOREM 4.2: *Suppose X has property **(A)**. Then $\mathcal{F}(X)$ is amenable.*

Proof: With notation as in the description of property **(A)**, define the net $(d_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{F}(X) \widehat{\otimes} \mathcal{F}(X)$ by

$$d_\lambda = \frac{1}{|\mathcal{G}_\lambda|} \sum_{g \in \mathcal{G}_\lambda} E_\lambda(g) \otimes E_\lambda(g^{-1}) \quad (\lambda \in \Lambda).$$

By assumption this is a bounded net. Observing that $\pi(d_\lambda) = P_\lambda$, we conclude by **A(i)** that $(\pi(d_\lambda))_{\lambda \in \Lambda}$ is a bounded left approximate identity for $\mathcal{F}(X)$.

Let $F \in \mathcal{F}(X)$. Then

$$\begin{aligned} F.d_\lambda - d_\lambda.F &= (F - P_\lambda F P_\lambda).d_\lambda - d_\lambda.(F - P_\lambda F P_\lambda) \\ &\quad + P_\lambda F P_\lambda.d_\lambda - d_\lambda.P_\lambda F P_\lambda \\ &= (F - P_\lambda F P_\lambda).d_\lambda - d_\lambda.(F - P_\lambda F P_\lambda), \end{aligned}$$

since $\frac{1}{|\mathcal{G}_\lambda|} \sum_{g \in \mathcal{G}_\lambda} g \otimes g^{-1}$ is a diagonal for $M_{n_\lambda}(\mathbb{C})$. By **A(ii)** (P_λ) is a bounded right approximate identity for $\mathcal{F}(X)$, so that $F.d_\lambda - d_\lambda.F \rightarrow 0$. ■

Remark 4.2.a: The condition of biorthogonality in property **(A)** is stronger than necessary. The following asymptotic trace condition suffices to establish amenability:

$$\frac{1}{n_\lambda} \sum_i^{n_\lambda} \langle x_{i,\lambda}, x_{i,\lambda}^* \rangle \rightarrow 1$$

along Λ . With this condition replacing biorthogonality all statements in this section about property **(A)** remain valid. We have made no use of this greater generality and so do not give the details here. However, if X has property **(A)**, then **A(i)** implies that X is a π -space and so probably there are spaces which satisfy the weaker condition but not property **(A)**. (Note that apparently there are spaces with the bounded approximation property which are not π -spaces, see the introduction to [C&K].)

Remark 4.2.b: Conditions **A(i)** and **A(ii)** together imply that X is what might be called a “shrinking π_λ -space”, compare with the discussion in [G&W]. Thus, if X has a basis and P_n is the projection onto the span of the first n basis elements, then (P_n) satisfying **A(i)** and **A(ii)** implies that the basis is a shrinking basis. If, furthermore, (P_n) satisfies **A(iii)** with the monomial group of degree n , then the basis is a symmetric basis, see [L&T, Ch. 3a]. In this case X will have property **(A)**, see also Theorem 4.5 below.

Property **(A)** is preserved for some natural Banach spaces formed from the original space, as set forth in the next two theorems. This will enable us to establish amenability of $\mathcal{F}(X)$ for a large class of Banach spaces, including all the classical spaces.

THEOREM 4.3: *Let X be a Banach space. If X^* has property **(A)**, then X has property **(A)**.*

Proof: Let

$$\{(x_{i,\lambda}^*, x_{i,\lambda}^{**})\} \quad (\lambda \in \Lambda)$$

be a net of biorthogonal systems satisfying the conditions of **(A)** with respect to X^* . Let \mathcal{U} and \mathcal{V} be the sets of all finite dimensional subspaces of X and X^* respectively, and let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be given. By means of the principle of local reflexivity [L&T], choose a linear map

$$S_{U,V,\lambda}: \text{span}(\{x_{i,\lambda}^{**} \mid i = 1, \dots, n_\lambda\} \cup U) \mapsto X$$

such that

- (1) $\|S_{U,V,\lambda}\| \leq 2$.
- (2) $S_{U,V,\lambda}|_U = 1_U$.
- (3) $\langle S_{U,V,\lambda}(x_{i,\lambda}^{**}), x^* \rangle = \langle x^*, x_{i,\lambda}^{**} \rangle$ for all $x^* \in \text{span}(\{x_{i,\lambda}^*\} \cup V)$.

We order \mathcal{U} and \mathcal{V} by containment and $\mathcal{U} \times \mathcal{V} \times \Lambda$ by the product order.

By construction

$$\{(S_{U,V,\lambda}x_{i,\lambda}^{**}, x_{j,\lambda}^*) \mid i, j = 1, \dots, n_\lambda\} \quad ((U, V, \lambda) \in \mathcal{U} \times \mathcal{V} \times \Lambda)$$

is a net of finite biorthogonal systems. We denote the corresponding lifts of matrix algebras by $E_{U,V,\lambda}$ and the corresponding projections by $P_{U,V,\lambda}$. Then

$$P_{U,V,\lambda} = S_{U,V,\lambda}P_\lambda^\alpha \iota_X,$$

where ι_X is the canonical inclusion of X into X^{**} and P_λ 's are the property **(A)** projections for X^* . Clearly $\{P_{U,V,\lambda}\}$ is a bounded set and for $x \in U$

$$\begin{aligned} \|P_{U,V,\lambda}x - x\| &= \|S_{U,V,\lambda}(P_\lambda^\alpha x) - x\| \\ &= \|S_{U,V,\lambda}(P_\lambda^\alpha x - x)\| \\ &\leq 2\|P_\lambda^\alpha x - x\|, \end{aligned}$$

where the two last steps follow from (1) and (2) above. Hence **A(i)** is satisfied. Similarly for $x^* \in V$

$$\begin{aligned} P_{U,V,\lambda}^\alpha(x^*) &= \sum x_{i,\lambda}^* \otimes S_{U,V,\lambda}(x_{i,\lambda}^{**})(x^*) \\ &= \sum \langle S_{U,V,\lambda}(x_{i,\lambda}^{**}), x^* \rangle x_{i,\lambda}^* \\ &= \sum \langle x^*, x_{i,\lambda}^{**} \rangle x_{i,\lambda}^* = P_\lambda^\alpha(x^*), \end{aligned}$$

using (3), so that A(ii) is satisfied. The supremum in A(iii) is increased by at most a factor 2, using the same irreducible matrix groups: $\mathcal{G}_{U,V,\lambda} = \mathcal{G}_\lambda$.

We have thus found a net of finite biorthogonal systems which satisfies the conditions needed for property (A). ■

Property (A) also behaves nicely with respect to tensor products:

THEOREM 4.4: *Let X and Y be Banach spaces and let Z be a tight tensor product of X and Y . If X and Y have property (A), then so does Z .*

Proof: We write $Z = X \otimes_\alpha Y$. Let $(\mathcal{O}_\lambda)_{\lambda \in \Lambda}$ and $(\mathcal{R}_\mu)_{\mu \in M}$ be property (A) nets of biorthogonal systems for X and Y respectively. We define the tensor product $(\mathcal{O}_\lambda \otimes \mathcal{R}_\mu)_{(\lambda,\mu) \in \Lambda \times M}$ to be the product ordered net of biorthogonal systems for $X \otimes_\alpha Y$ given as

$$\mathcal{O}_\lambda \otimes \mathcal{R}_\mu = \{(x \otimes y, x^* \otimes y^*) \mid (x, x^*) \in \mathcal{O}_\lambda, (y, y^*) \in \mathcal{R}_\mu\}.$$

Using the identification $M_n(\mathbb{C}) \otimes M_p(\mathbb{C}) = M_{np}(\mathbb{C})$, one checks easily that the property (A) lifts

$$E_{(\lambda,\mu)}: M_{n_\lambda n_\mu}(\mathbb{C}) \rightarrow \mathcal{F}(Z)$$

are nothing but $E_{(\lambda,\mu)} = C_\lambda \otimes D_\mu$, where C_λ and D_μ are the lifts belonging to X and Y respectively. Hence A(i) holds for Z and, since $X^* \otimes Y^*$ is dense in Z^* , we also have A(ii). To obtain A(iii) it suffices to notice that, if \mathcal{G} and \mathcal{H} are irreducible $(m \times m)$ - and $(n \times n)$ -matrix groups, then $\mathcal{G} \otimes \mathcal{H}$ is an irreducible $(mn \times mn)$ -matrix group. ■

We shall now give some concrete examples of spaces with property (A). The first is very much in the spirit of [J, Proposition 6.1]. Recall that a basis $(x_n)_{n \in \mathbb{N}}$ in a Banach space X is called **subsymmetric** if $(x_n)_{n \in \mathbb{N}}$ is unconditional and equivalent to the basis sequence $(x_{n_i})_{i \in \mathbb{N}}$ for every increasing sequence $(n_i)_{i \in \mathbb{N}}$, see [L&T, Ch. 3.a] and [Si, Ch. 21].

THEOREM 4.5: *Suppose that X has a subsymmetric and shrinking basis. Then X has property (A).*

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be a subsymmetric and shrinking basis and let $(x_n^*)_{n \in \mathbb{N}}$ be the associated sequence of coordinate functionals. Then

$$\{(x_i, x_j^*) \mid i, j = 1, \dots, n\} \quad (n \in \mathbb{N})$$

is a sequence of finite biorthogonal systems satisfying the conditions of property **(A)**. **A(i)** is immediate, **A(ii)** follows from the basis being shrinking. To prove **A(iii)** we shall use the following observations.

Since $(x_n)_{n \in \mathbb{N}}$ is unconditional, the family of operators of the form

$$(4.1) \quad U \left(\sum_{n \in \mathbb{N}} a_n x_n \right) = \sum_{n \in \eta} s(n) a_n x_n,$$

where $\eta \subseteq \mathbb{N}$ and $s \in \{\pm 1\}^{\mathbb{N}}$, is uniformly bounded, say by $K > 0$. The subsymmetry means that the family of operators of the form

$$(4.2) \quad A_{(m_i)(n_i)}(x) = \sum_{i=1}^{\infty} x_{m_i}^*(x) x_{n_i},$$

is uniformly bounded, say by $M > 0$. Here $(m_i)_{i \in \mathbb{N}}$ and $(n_i)_{i \in \mathbb{N}}$ are two arbitrary increasing sequences of integers.

Let \mathcal{G}_n be the subgroup of the monomial group of degree n defined by the permutation matrix σ corresponding to the cyclic permutation $(12 \cdots n)$, i.e.

$$\mathcal{G}_n = \{D(t)\sigma^k \mid t \in \{\pm 1\}^n, k = 0, \dots, n - 1\}.$$

By Lemma 3.1 \mathcal{G}_n is an irreducible $(n \times n)$ -matrix group. We write elements in X as sequences. Then for $g = D(t)\sigma^k$ we have:

$$E(g)(\xi_1, \xi_2, \dots, \xi_n, \dots) = (t(1)\xi_{n+1-k}, \dots, t(k)\xi_n, t(k+1)\xi_1, \dots, t(n)\xi_{n-k}, 0, \dots).$$

We see that $E(g)$ has the form $E(g) = A_1 U_1 + A_2 U_2$ for appropriate choices of operators U_i of type (4.1) and A_i of type (4.2). Hence the supremum in **A(iii)** does not exceed $2KM$. ■

COROLLARY 4.6: *Let X be a reflexive Orlicz sequence space or a reflexive Lorentz sequence space. Then X has property **(A)** and so $\mathcal{F}(X)$ is amenable.*

Proof: See [L&T, Ch. 4.a] for a discussion showing that these spaces satisfy the hypotheses of Theorem 4.5. ■

Remark 4.6.a: Actually the spaces of Corollary 4.6 have symmetric bases, so the proof of [J, Proposition 6.1] could easily be adapted to give amenability of $\mathcal{F}(X)$

in these cases. However, there are spaces with subsymmetric, non-symmetric bases, see for instance [Ga, § 5].

We shall now give substance to the remark that the setup of property (A) is customized to deal with the classical spaces.

THEOREM 4.7: *Let K be a compact Hausdorff space and let (Ω, Σ, μ) be a measure space. Then $C(K)$ and $L_p(\mu)$, $1 \leq p \leq \infty$, have property (A).*

Proof: Since $C(K)^* = L_1(\mu_K)$ for a suitable measure space $(\Omega_K, \Sigma_K, \mu_K)$ and $L_\infty(\mu) = C(K_\mu)$ for a suitable compact space K_μ , it follows from Theorem 4.3 that it is enough to consider $L_p(\mu)$ for $1 \leq p < \infty$. We shall give the proof in detail in the case of a probability space, cf. the remark below. Let \mathcal{S} be a finite collection of disjoint measurable subsets of Ω whose union is all of Ω . As it is customary in integration theory we order such dissections by $\mathcal{S}_1 \prec \mathcal{S}_2$ if every set in \mathcal{S}_1 is a union of sets from \mathcal{S}_2 . We define the biorthogonal systems in $L_p(\mu) \times L_{p'}(\mu)$ $\frac{1}{p} + \frac{1}{p'} = 1$ by

$$\mathcal{O}_{\mathcal{S}} = \{(\chi_L, \chi'_M) \mid L, M \in \mathcal{S}\},$$

where χ_\bullet and χ'_\bullet denote indicator functions normalized in $L_p(\mu)$ and $L_{p'}(\mu)$ respectively. It is now a routine matter to verify property (A). Let $P_{\mathcal{S}}$ be the property (A) projections. For a measurable set $M \in \Sigma$ we have

$$(4.3) \quad \begin{aligned} P_{\mathcal{S}}(\chi_M) &= \chi_M, \\ P_{\mathcal{S}}^a(\chi'_M) &= \chi'_M, \end{aligned}$$

whenever $\{M\} \prec \mathcal{S}$, so A(i) and A(ii) are immediate. To prove A(iii), consider $\mathcal{S} = \{M_1, \dots, M_n\}$ and define $\mathcal{G}_{\mathcal{S}}$ to be the monomial group of degree n . Let $g = D(t)\sigma$ where $t \in \{\pm 1\}^n$ and σ is a permutation matrix. First notice that

$$\left\| \sum_{i=1}^n a_i \chi_{M_i} \right\|^p = \sum_{i=1}^n |a_i|^p.$$

Using this we get for an arbitrary $f \in L_p(\mu)$

$$\begin{aligned} \|E_S(g)f\|^p &= \left\| \sum_{i=1}^n t(i)\langle f, \chi'_{M_i} \rangle \chi_{M_{\sigma(i)}} \right\|^p \\ &= \sum_{i=1}^n |\langle f, \chi'_{M_i} \rangle|^p \\ &\leq \sum_{i=1}^n \int_{M_i} |f|^p d\mu \quad (\text{H\"older Inequality}) \\ &= \sum_{i=1}^n \int_{M_i} |f|^p d\mu \\ &= \|f\|^p. \quad \blacksquare \end{aligned}$$

Remark 4.7.a: A proof of the general case can be given along the same lines but with added minor technicalities. Alternatively, we may reduce it to the special case. We are interested only in finite-dimensional subspaces. Functions in such a subspace are supported on a σ -finite measure space. The corresponding complemented subspaces of $L_p(\mu)$ have projection constants uniformly bounded by 1 and are isometrically isomorphic to L_p -spaces of probability measures.

Remark 4.7.b: The proof also works for a rearrangement invariant order continuous function space, X , say on one of the intervals $[0, 1]$ or $[0, \infty)$, for instance separable Orlicz or Lorentz functions spaces, see [L&T II, p. 120]. We are grateful to the referee for having brought this to our attention. To modify the proof we just have to choose the sets in S of equal measure. Since X is order continuous (4.3) will, as above, ensure that **A**(i) and **A**(ii) hold. In the estimate of $\|E_S(g)f\|$ we get $\|E_S(g)f\| \leq \|P_S f\|$. A simple interpolation argument using [L&T II, Theorem 2.a.10] then shows that **A**(iii) holds.

Combining Theorem 4.7 with Theorem 2.5 and Theorem 4.4 we get a large collection of Banach spaces carrying amenable algebras.

COROLLARY 4.8: *Suppose that X has property **(A)**. If X^* has RNP, then $C(K, X)$ has property **(A)**. If X^* has RNP with respect to μ , then $L_p(\mu, X)$ has property **(A)** for $1 \leq p \leq \infty$.*

COROLLARY 4.9: *$\mathcal{F}(\ell_p \widehat{\otimes} \ell_q)$ is amenable if and only if $1/p + 1/q < 1$.*

Proof: By Corollary 2.6 $\mathcal{F}(\ell_p \widehat{\otimes} \ell_q)$ is amenable for $1/p + 1/q < 1$. In [A&F] it is shown that, if $r \leq s$, then $\mathcal{B}(\ell_r, \ell_s) = (\ell_r \widehat{\otimes} \ell_s)^*$ contains a complemented

copy of $\mathcal{B}(\ell_2)$ and thus fails the approximation property, [Sz]. Hence, when $1/p+1/q \geq 1$, then $\mathcal{F}(\ell_p \widehat{\otimes} \ell_q)$ does not have a bounded right approximate identity and is consequently not amenable. ■

Probably it is too much to hope that amenability of $\mathcal{F}(X)$ is equivalent to X having property (A). Since the approximate diagonal stemming from property (A) is obtained by means of lifts of the canonical diagonals of matrix algebras, it will have the approximate versions of the extra properties (3.3) and (3.4). It seems unlikely that such approximate diagonals should always exist, once amenability of $\mathcal{F}(X)$ is established. In Section 6 we will see examples of spaces for which $\mathcal{F}(X)$ is amenable but for which we do not know whether X has property (A) or even the weaker property mentioned in Remark 4.2.a.

5. Dual spaces

We have seen that property (A) passes from a dual Banach space to its predual. The following stability property for amenability implies an extension of this fact, namely, that if the algebra of approximable operators on a dual space is amenable, then the algebra of approximable operators on any predual of the space is amenable. It also implies a similar, but weaker, result for the algebra of compact operators.

THEOREM 5.1: *Let \mathcal{A} be an amenable Banach algebra and \mathcal{I} be a closed, left ideal in \mathcal{A} which has a bounded two-sided approximate identity. Then \mathcal{I} is amenable.*

Proof: It is convenient to define a new product on $\mathcal{A} \widehat{\otimes} \mathcal{A}$ by $(a \otimes b) \bullet (c \otimes d) = ac \otimes bd$, that is, \bullet is the usual product on $\mathcal{A} \widehat{\otimes} \mathcal{A}$. Let $(d_\alpha)_{\alpha \in A}$ be an approximate diagonal for \mathcal{A} , let $(e_\beta)_{\beta \in B}$ be a bounded two-sided approximate identity for \mathcal{I} , and put

$$p_{\alpha\beta\gamma} = d_\alpha \bullet (e_\beta \otimes e_\gamma) \quad (\alpha \in A; \beta, \gamma \in B).$$

Then, since \mathcal{I} is a left ideal and $(d_\alpha)_{\alpha \in A}$ and $(e_\beta)_{\beta \in B}$ are bounded nets, $p_{\alpha\beta\gamma}$ belongs to a bounded subset of $\mathcal{I} \widehat{\otimes} \mathcal{I}$.

For each c in \mathcal{I} we have

$$\begin{aligned} & \limsup_{\gamma} \|c \cdot p_{\alpha\beta\gamma} - p_{\alpha\beta\gamma} \cdot c\| \\ &= \limsup_{\gamma} \|(c \otimes 1) \bullet d_{\alpha} \bullet (e_{\beta} \otimes e_{\gamma}) - d_{\alpha} \bullet (e_{\beta} \otimes e_{\gamma}) \bullet (1 \otimes c)\| \\ &= \limsup_{\gamma} \|((c \otimes 1) \bullet d_{\alpha} - d_{\alpha} \bullet (1 \otimes c)) \bullet (e_{\beta} \otimes e_{\gamma})\| \\ &= \limsup_{\gamma} \|(c \cdot d_{\alpha} - d_{\alpha} \cdot c) \bullet (e_{\beta} \otimes e_{\gamma})\|, \end{aligned}$$

using $\lim_{\gamma} (e_{\gamma}c - ce_{\gamma}) = 0$.

Since $(d_{\alpha})_{\alpha \in A}$ is an approximate diagonal and $(e_{\beta})_{\beta \in B}$ is bounded we get from the inequality

$$\|(cd_{\alpha} - d_{\alpha}c) \bullet e_{\beta} \otimes e_{\gamma}\| \leq \|(cd_{\alpha} - d_{\alpha}c)\| \|e_{\beta}\| \|e_{\gamma}\|$$

that

$$\lim_{\alpha} \limsup_{\beta} \limsup_{\gamma} \|(cd_{\alpha} - d_{\alpha}c) \bullet e_{\beta} \otimes e_{\gamma}\| = 0,$$

and so

$$\lim_{\alpha} \limsup_{\beta} \limsup_{\gamma} \|c \cdot p_{\alpha\beta\gamma} - p_{\alpha\beta\gamma} \cdot c\| = 0.$$

Furthermore,

$$\begin{aligned} \lim_{\alpha} \lim_{\beta} \lim_{\gamma} \pi(p_{\alpha\beta\gamma})c &= \lim_{\alpha} \lim_{\beta} \lim_{\gamma} \pi(d_{\alpha} \bullet (e_{\beta} \otimes e_{\gamma}))c \\ &= \lim_{\alpha} \lim_{\beta} \pi(d_{\alpha} \bullet (e_{\beta} \otimes 1))c \\ &= \lim_{\alpha} \pi(d_{\alpha})c \\ &= c, \end{aligned}$$

where the second and third equality follow from $(e_{\beta})_{\beta \in B}$ being a left approximate identity for \mathcal{I} and \mathcal{I} being a left ideal, and the last from $(d_{\alpha})_{\alpha \in A}$ being an approximate diagonal. It follows that we may choose a net from $\{p_{\alpha\beta\gamma} \mid \alpha \in A; \beta, \gamma \in B\}$ which is an approximate diagonal for \mathcal{I} . Therefore \mathcal{I} is amenable. ■

This theorem is an improvement on the last assertion in Proposition 5.1 in [J]. It may also be shown, by a similar argument but with $d_{\alpha}(e_{\beta} \otimes e_{\gamma})$ in place of $d_{\alpha} \bullet (e_{\beta} \otimes e_{\gamma})$, that, if \mathcal{A} is an amenable Banach algebra and \mathcal{I} is a two-sided ideal in \mathcal{A} with a bounded left approximate identity, then \mathcal{I} is amenable.

COROLLARY 5.2: *Let X be a Banach space such that $\mathcal{K}(X^*)$ is amenable and $\mathcal{K}(X)$ has a bounded two-sided approximate identity. Then $\mathcal{K}(X)$ is amenable.*

Proof: $\mathcal{K}(X)^\alpha$, which is anti-isomorphic to $\mathcal{K}(X)$, is a closed left ideal in $\mathcal{K}(X^*)$ and has a bounded two-sided approximate identity. ■

Example 4.3 in [G&W] provides a Banach space, X , such that $\mathcal{K}(X^*)$ has a bounded two-sided approximate identity but $\mathcal{K}(X)$ does not. This example suggests that the hypothesis that $\mathcal{K}(X)$ has a bounded two-sided approximate identity is necessary. However, if X has the approximation property it is not.

COROLLARY 5.3: *Let X be a Banach space such that $\mathcal{F}(X^*)$ is amenable. Then $\mathcal{F}(X)$ is amenable.*

Proof: Since $\mathcal{F}(X^*)$ has a bounded left approximate identity, X^* has the bounded approximation property, by [D, Theorem 2.6]. Hence, by [G&W, Theorem 3.3], $\mathcal{F}(X)$ has a bounded two-sided approximate identity. ■

It is an open question, which is discussed further in Section 7, whether amenability of $\mathcal{K}(X)$ implies that X has the approximation property.

The converse to Corollary 5.2 holds if $\mathcal{K}(X^*)$ has a bounded two-sided approximate identity. This fact will follow from another stability property of amenability.

THEOREM 5.4: *Let \mathcal{A} be a Banach algebra which has a bounded two-sided approximate identity and let \mathcal{I} be a closed, left ideal in \mathcal{A} which is amenable and has a bounded left approximate identity for \mathcal{A} . Then \mathcal{A} is amenable.*

Proof: By Proposition 1.8 in [J], it will suffice to check that all derivations from \mathcal{A} into duals of essential \mathcal{A} -bimodules are inner. (An \mathcal{A} -bimodule Y is **essential** if $Y = \text{span}\{a.y.b : a, b \in \mathcal{A}; y \in Y\}$, because, with the hypothesis of a bounded approximate identity, this last space is closed.)

Let $D: \mathcal{A} \rightarrow Y^*$ be a derivation, where Y is an essential \mathcal{A} -bimodule. Since \mathcal{I} is amenable, there is y^* in Y^* such that $Da = a.y^* - y^*.a$ for every a in \mathcal{I} . Then the map, $\delta: \mathcal{A} \rightarrow Y^*$, defined by $\delta a = a.y^* - y^*.a$ is an inner derivation from \mathcal{A} and so $D - \delta$ is a derivation from \mathcal{A} whose restriction to \mathcal{I} is zero.

Now let a and b belong to \mathcal{A} and let $(e_\lambda)_{\lambda \in \Lambda}$ be a bounded net in \mathcal{I} which

is a left approximate identity for \mathcal{A} . Then, since \mathcal{I} is a left ideal,

$$\begin{aligned} 0 &= \lim_{\lambda} (D - \delta)(ae_{\lambda}).b \\ &= \lim_{\lambda} (D - \delta)(a).e_{\lambda}b \\ &= (D - \delta)(a).b, \end{aligned}$$

where the two first identities are true because $D - \delta$ is a derivation which annihilates \mathcal{I} , and the third because $(e_{\lambda})_{\lambda \in \Lambda}$ is a left approximate identity.

It follows that $\langle b.y, (D - \delta)(a) \rangle = 0$ for every y in Y and a and b in \mathcal{A} . Since Y is essential, $D = \delta$ and is thus inner. ■

The next result may now be proved in a similar way to Corollary 5.2.

COROLLARY 5.5: *Let X be a Banach space such that $\mathcal{K}(X)$ is amenable and $\mathcal{K}(X^*)$ has a bounded two-sided approximate identity. Then $\mathcal{K}(X^*)$ is amenable.*

The argument of Proposition 6.1 in [J] shows, without change, that $\mathcal{K}(c_0)$ is amenable. It follows from this corollary and the fact that $\mathcal{K}(\ell_1)$ has a bounded two-sided approximate identity that $\mathcal{K}(\ell_1)$ is amenable. Proposition 6.1 in [J] does not yield this fact about ℓ_1 directly, although we have shown it in Section 2 by modifying the argument in [J] suitably.

Example 5.6: The requirement in Corollary 5.5 that $\mathcal{K}(X^*)$ have a bounded two-sided approximate identity is necessary. Since ℓ_2 is reflexive it has the RNP. Hence, by Theorem 2.5, $\ell_2 \overset{\vee}{\otimes} \ell_2$ is a tight tensor product and so, by Theorem 2.2, $\mathcal{F}(\ell_2 \overset{\vee}{\otimes} \ell_2)$ is amenable. Now $(\ell_2 \overset{\vee}{\otimes} \ell_2)^*$ is isomorphic to $\ell_2 \widehat{\otimes} \ell_2$ and $(\ell_2 \overset{\vee}{\otimes} \ell_2)^{**}$ to $\mathcal{B}(\ell_2)$. Since $\mathcal{F}(\ell_2 \overset{\vee}{\otimes} \ell_2)$ has a bounded two-sided approximate identity, $\mathcal{F}(\ell_2 \widehat{\otimes} \ell_2)$ has a bounded left approximate identity, see [G&W, Theorem 3.3]. However, $\mathcal{B}(H)$ does not have the approximation property (see [Sz]) and so $\mathcal{F}(\ell_2 \widehat{\otimes} \ell_2)$ does not have a bounded right approximate identity. Therefore, $\mathcal{F}((\ell_2 \overset{\vee}{\otimes} \ell_2)^*)$ is not amenable.

6. Direct sums

In the following it is necessary to use the algebra of double multipliers on a Banach algebra \mathcal{A} . A double multiplier on \mathcal{A} is a pair of bounded operators, (L, R) , on \mathcal{A} which commute and satisfy, for all a and b in \mathcal{A} : $L(ab) = L(a)b$; $R(ab) = aR(b)$; and $aL(b) = R(a)b$. Denote the set of all double multipliers on \mathcal{A} by $M(\mathcal{A})$. Then

$M(\mathcal{A})$ is a Banach space with the obvious norm and sum and becomes a Banach algebra when equipped with the product $(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1)$. If $T = (L, R)$ is a double multiplier on \mathcal{A} , then $L(a)$ will be denoted by Ta and $R(a)$ by aT .

Each element, a , of \mathcal{A} determines a double multiplier, (L_a, R_a) , where L_a and R_a are respectively the operators on \mathcal{A} of left and right multiplication by a . Similarly, if \mathcal{A} is embedded as an ideal in a Banach algebra \mathcal{B} , then each element of \mathcal{B} determines a double multiplier on \mathcal{A} . Thus each operator on the Banach space X determines a double multiplier on $\mathcal{K}(X)$ and on $\mathcal{F}(X)$. Note also that there is always an identity, I , in $M(\mathcal{A})$.

Now let P_1 be an idempotent in $M(\mathcal{A})$ and put $P_2 = I - P_1$ and $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, $i, j = 1, 2$. Next put $\mathcal{A}_{11}^\circ = \pi(\mathcal{A}_{12}\widehat{\otimes}\mathcal{A}_{21})$ and $\mathcal{A}_{22}^\circ = \pi(\mathcal{A}_{21}\widehat{\otimes}\mathcal{A}_{12})$, where π denotes the product in \mathcal{A} . Then \mathcal{A}_{ii}° is isomorphic, as a linear space, to the quotient of $\mathcal{A}_{ij}\widehat{\otimes}\mathcal{A}_{ji}$, $j \neq i$, by $\ker(\pi) \cap (\mathcal{A}_{ij}\widehat{\otimes}\mathcal{A}_{ji})$. Let $\|\cdot\|^\circ$ denote the quotient norm on \mathcal{A}_{ii}° .

In this section we prove a couple of abstract results about the stability of amenability when \mathcal{A} is cut down to \mathcal{A}_{11} by an idempotent in $M(\mathcal{A})$ and then apply them to the case where $\mathcal{A} = \mathcal{K}(X)$ for some Banach space X and P_1 is determined by a projection on X . We will thus establish some stability properties of amenability of $\mathcal{K}(X)$ under direct sums of Banach spaces.

PROPOSITION 6.1: *Let \mathcal{A} and \mathcal{A}_{ij} , $i, j = 1, 2$, be as above. Then \mathcal{A} has a bounded two-sided approximate identity if and only if \mathcal{A}_{11} and \mathcal{A}_{22} have bounded two-sided approximate identities and \mathcal{A}_{ij} is an essential left \mathcal{A}_{ii} - and right \mathcal{A}_{jj} -module, $i, j = 1, 2$.*

Proof: Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a bounded net in \mathcal{A} . Then $\{P_1e_\lambda P_1 + P_2e_\lambda P_2\}_{\lambda \in \Lambda}$ is a two-sided approximate identity if and only if $\{P_i e_\lambda P_i\}_{\lambda \in \Lambda}$ is a two-sided approximate identity in \mathcal{A}_{ii} , a left approximate identity for \mathcal{A}_{ij} and a right approximate identity for \mathcal{A}_{ji} , $i = 1, 2$; $j \neq i$. ■

The first of the abstract results is the following

THEOREM 6.2: *Let \mathcal{A} and \mathcal{A}_{ij} , $i, j = 1, 2$, be as above and suppose that \mathcal{A} has a bounded two-sided approximate identity and that $\mathcal{A}_{22} = \mathcal{A}_{22}^\circ$. Then \mathcal{A} is amenable if and only if \mathcal{A}_{11} is amenable.*

Proof: The inclusion map $\mathcal{A}_{22}^\circ \rightarrow \mathcal{A}_{22}$ is continuous and is also a surjection. Hence, by the open mapping theorem, $\|\cdot\|^\circ$ is equivalent to the given norm on \mathcal{A}_{22} .

Furthermore, since \mathcal{A} has a bounded two-sided approximate identity, Proposition 6.1 shows that \mathcal{A}_{22} also has a bounded two-sided approximate identity. Therefore there is a bounded net $\{c^\beta\}_{\beta \in B}$ in $\mathcal{A}_{21} \widehat{\otimes} \mathcal{A}_{12}$ such that $\{\pi(c^\beta)\}_{\beta \in B}$ is a bounded approximate identity for \mathcal{A}_{22} . The elements of this net have the form $c^\beta = \sum_i r_i^\beta \otimes s_i^\beta$.

Now suppose that \mathcal{A}_{11} is amenable and let $\{d_{11}^\alpha\}_{\alpha \in A}$ be an approximate diagonal for \mathcal{A}_{11} . We will show that \mathcal{A} is amenable by showing that it has an approximate diagonal consisting of elements of the form

$$d^{\alpha,\beta} = d_{11}^\alpha + c_\beta d_{11}^\alpha.$$

Here we have equipped $\mathcal{A} \widehat{\otimes} \mathcal{A}$ with the product $(a \otimes b)(c \otimes d) = ac \otimes db$ as described in the introduction. Note first of all that the set of all such elements is bounded because $\|d^{\alpha,\beta}\| \leq \|d_{11}^\alpha\|(1 + \|c^\beta\|)$. In order to prove that an approximate diagonal can be constructed, we shall use the following

$$(6.1) \quad \begin{aligned} & \lim_\alpha [(\pi(c) \otimes 1)(a_{21} \otimes 1) - (1 \otimes a_{21})c]d_{11}^\alpha \\ &= \lim_\alpha [(a_{12} \otimes 1)c - (1 \otimes \pi(c))(1 \otimes a_{12})]d_{11}^\alpha = 0, \end{aligned}$$

for each $c \in \mathcal{A}_{21} \widehat{\otimes} \mathcal{A}_{12}$ and $a_{ij} \in \mathcal{A}_{ij}$. It is enough to prove (6.1) for c an elementary tensor $b_{21} \otimes b_{12}$. Then the first expression equals $(b_{21} \otimes 1)(b_{12}a_{21} \otimes 1 - 1 \otimes b_{12}a_{21})d_{11}^\alpha$, which tends to zero, because (d_{11}^α) is an approximate diagonal for \mathcal{A}_{11} . The other limit is obtained analogously.

We will show that

$$(6.2) \quad \lim_\beta \lim_\alpha \pi(d^{\alpha,\beta})a = a \quad (a \in \mathcal{A}),$$

and

$$(6.3) \quad \lim_\beta \limsup_\alpha \|(a \otimes 1 - 1 \otimes a)d^{\alpha,\beta}\| = 0 \quad (a \in \mathcal{A}).$$

This will imply that an approximate diagonal can be constructed from the $d^{\alpha,\beta}$'s.

First we prove (6.2). If a is in \mathcal{A} , then $a = P_1a + P_2a$ and (6.2) follows because we have

$$\lim_\alpha \pi(d^{\alpha,\beta})P_1a = \lim_\alpha \pi(d_{11}^\alpha)P_1a = P_1a,$$

since $\pi(c_\beta d_{11}^\alpha) \in \mathcal{A}_{21}\pi(d_{11}^\alpha)\mathcal{A}_{12}$ and $\pi(d_{11}^\alpha)$ is a bounded approximate identity for \mathcal{A}_{11} . Likewise

$$\begin{aligned} \lim_{\beta} \lim_{\alpha} \pi(d^{\alpha,\beta})P_2a &= \lim_{\beta} \lim_{\alpha} \pi(c_\beta d_{11}^\alpha)P_2a \\ &= \lim_{\beta} \pi(c_\beta)P_2a = P_2a, \end{aligned}$$

again since $\pi(c_\beta d_{11}^\alpha) \in \mathcal{A}_{21}\pi(d_{11}^\alpha)\mathcal{A}_{12}$.

Now we prove (6.3). Since $a = a_{11} + a_{12} + a_{21} + a_{22}$, where a_{ij} is in \mathcal{A}_{ij} , we may treat these terms separately. We have

$$\begin{aligned} (a_{11} \otimes 1 - 1 \otimes a_{11})d^{\alpha,\beta} &= (a_{11} \otimes 1 - 1 \otimes a_{11})d_{11}^\alpha, \\ (a_{12} \otimes 1 - 1 \otimes a_{12})d^{\alpha,\beta} &= (a_{12} \otimes 1)c_\beta d_{11}^\alpha - (1 \otimes a_{12})d_{11}^\alpha, \\ (a_{21} \otimes 1 - 1 \otimes a_{21})d^{\alpha,\beta} &= (a_{21} \otimes 1)d_{11}^\alpha - (1 \otimes a_{21})c_\beta d_{11}^\alpha, \\ (a_{22} \otimes 1 - 1 \otimes a_{22})d^{\alpha,\beta} &= (a_{22} \otimes 1 - 1 \otimes a_{22})c_\beta d^{\alpha,\beta}. \end{aligned}$$

Clearly the first term tends to 0 as $\alpha \rightarrow \infty$. The second term may be rewritten as

$$((a_{12} \otimes 1)c_\beta - (1 \otimes \pi(c_\beta))(1 \otimes a_{12}))d_{11}^\alpha + 1 \otimes (a_{12}\pi(c_\beta) - a_{12})d_{11}^\alpha$$

so that, using (6.1) and that $(\pi(c_\beta))_{\beta \in B}$ is a bounded right approximate identity for \mathcal{A}_{12} , the statement (6.3) is true in this case.

The third term may be rewritten as

$$((\pi(c_\beta) \otimes 1)(a_{21} \otimes 1) - (1 \otimes a_{21})c_\beta)d_{11}^\alpha + ((a_{21} - \pi(c_\beta)a_{21}) \otimes 1)d_{11}^\alpha$$

and treated analogously.

For the fourth term it is enough to look at elements of the form $a_{22} = b_{21}b_{12}$ since by assumption these elements span a dense subset of \mathcal{A}_{22} and we are working with bounded nets. We then get

$$\begin{aligned} (a_{22} \otimes 1 - 1 \otimes a_{22})d^{\alpha,\beta} \\ = (b_{21} \otimes 1)(b_{12} \otimes 1 - 1 \otimes b_{12})d^{\alpha,\beta} + (1 \otimes b_{12})(b_{21} \otimes 1 - 1 \otimes b_{21})d^{\alpha,\beta}, \end{aligned}$$

so that this case follows from the two previous cases.

To prove the converse, suppose now that \mathcal{A} is amenable and let $\{d^\alpha\}_{\alpha \in A}$ be an approximate diagonal for \mathcal{A} . Using the multiplier multiplication $(P_i \otimes P_j)(a \otimes b) = P_i a \otimes b P_j$ and $(a \otimes b)(P_i \otimes P_j) = a P_i \otimes P_j b$, we define

$$d_{11}^{\alpha, \beta} = (P_1 \otimes P_1) d^\alpha (P_1 \otimes P_1 + c_\beta) \quad (\alpha \in A, \beta \in B).$$

First note that for an elementary tensor we have

$$\begin{aligned} \lim_{\beta} \pi((a \otimes b) c_\beta) &= \lim_{\beta} \pi((a \otimes b)(P_2 \otimes P_2) c_\beta) \\ &= \lim_{\beta} a P_2 \pi(c_\beta) P_2 a \\ &= a P_2 b \\ &= \pi((a \otimes b)(P_2 \otimes P_2)) \end{aligned}$$

so that

$$\begin{aligned} \lim_{\beta} \pi(d_{11}^{\alpha, \beta}) &= P_1 \lim_{\beta} \pi(d^\alpha (P_1 \otimes P_1) + c_\beta) P_1 \\ &= P_1 \pi(d^\alpha (P_1 \otimes P_1 + P_2 \otimes P_2)) P_1 \\ &= P_1 \pi(d^\alpha) P_1, \end{aligned}$$

which is a bounded left approximate identity for \mathcal{A}_{11} , directed over $\alpha \in A$.

For $a_{11} \in \mathcal{A}_{11}$ we have

$$\begin{aligned} (a_{11} \otimes 1 - 1 \otimes a_{11}) d_{11}^{\alpha, \beta} &= (a_{11} \otimes 1 - 1 \otimes a_{11})(P_1 \otimes P_1) d^\alpha (P_1 \otimes P_1 + c_\beta) \\ &= (P_1 \otimes P_1)(a_{11} \otimes 1 - 1 \otimes a_{11}) d^\alpha (P_1 \otimes P_1 + c_\beta), \end{aligned}$$

which tends to 0 as $\alpha \rightarrow \infty$, because $(d^\alpha)_{\alpha \in A}$ is an approximate diagonal for \mathcal{A} . This concludes the proof of the theorem. ■

We now give some applications of this theorem in the case when $\mathcal{A} = \mathcal{K}(X)$.

THEOREM 6.3: *Let X be a Banach space. Then $\mathcal{K}(X)$ is amenable if and only if $\mathcal{K}(X \oplus \mathbb{C})$ is amenable.*

Proof: Let $\mathcal{A} = \mathcal{K}(X \oplus \mathbb{C})$ and P_1 be the projection of $X \oplus \mathbb{C}$ onto X with kernel \mathbb{C} . Then $P_2 = I - P_1$ is the rank one projection onto \mathbb{C} with kernel X . Hence $\mathcal{A}_{22} = P_2 \mathcal{K}(X \oplus \mathbb{C}) P_2$ is the one-dimensional algebra spanned by P_2 .

It is easily seen that $\mathcal{A}_{22} = \mathcal{A}_{22}^\circ$. Furthermore, since \mathcal{A}_{22} has an identity, \mathcal{A} has a bounded two-sided approximate identity if either \mathcal{A} or \mathcal{A}_{11} is amenable. Theorem 6.2 now applies. ■

Many of the classical Banach spaces are isomorphic to their direct sum with the one-dimensional space and are also isomorphic to their hyperplanes. For some time it was an unsolved problem, the so-called ‘hyperplane problem’, whether every Banach space has this property. However, it is now known ([G&M]) that there is a Banach space which is not isomorphic to any proper subspace and so the above theorem has some content.

An important class of Banach spaces is the class of \mathcal{L}_p -spaces, where $1 \leq p \leq \infty$, which were introduced in [L&P]. The Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ -space if there is a constant $\lambda > 0$ such that for every finite dimensional subspace, B , of X there is a finite dimensional subspace, C , of X such that $B \subseteq C$ and $d(C, \ell_p^n) \leq \lambda$, where $n = \dim C$. (If Y and Z are isomorphic Banach spaces, then $d(Y, Z)$ is $\inf(\|T\|, \|T^{-1}\|)$, where the infimum is over all invertible operators, T , from Y onto Z .) Some examples of \mathcal{L}_p -spaces are ℓ_p and $L_p(0, 1)$. We have already seen in Theorem 4.7 that the algebras of compact operators on these examples are amenable.

Theorem III(c) in [L&R] shows that \mathcal{L}_p -spaces satisfy stronger conditions than they are defined to have. Thus, if X is an \mathcal{L}_p -space, then there is a constant $\lambda' > 0$ such that for every finite dimensional subspace, B , of X there are a finite dimensional subspace, C , of X and a projection, P , of X onto C such that: $B \subseteq C$, $d(C, \ell_p^n) \leq \lambda'$, where $n = \dim C$; and $\|P\| < \lambda'$. It follows that every \mathcal{L}_p -space has the approximation property, and so $\mathcal{K}(X) = \mathcal{F}(X)$ whenever X is an \mathcal{L}_p -space. It follows also that, if X and Y are infinite dimensional \mathcal{L}_p -spaces, then every T in $\mathcal{F}(X)$ is a product $T = UV$, where $U: X \rightarrow Y$ and $V: Y \rightarrow X$ are compact operators. Furthermore, if X is an \mathcal{L}_p -space, then X^* is an \mathcal{L}_q -space, where $q^{-1} + p^{-1} = 1$ ([L&R, Theorem III(a)]). Hence X^* has the bounded approximation property and so $\mathcal{F}(X)$ has a bounded two-sided approximate identity ([G&W, Theorem 3.3]). We are now ready to prove

THEOREM 6.4: *Let $1 \leq p \leq \infty$ and let X be an \mathcal{L}_p -space. Then $\mathcal{F}(X)$ is amenable.*

Proof: Let $\mathcal{A} = \mathcal{F}(\ell_p \oplus X)$ and let P_1 be the idempotent in $M(\mathcal{A})$ determined by the projection onto ℓ_p with kernel X . Then \mathcal{A} has a bounded two-sided approximate identity because $\mathcal{A}_{11} = \mathcal{F}(\ell_p)$ and $\mathcal{A}_{22} = \mathcal{F}(X)$ do. Also, since each compact operator on X factors through ℓ_p , $\mathcal{A}_{22}^\circ = \mathcal{A}_{22}$. Therefore, since $\mathcal{F}(\ell_p)$ is amenable, $\mathcal{F}(\ell_p \oplus X)$ is amenable by Theorem 6.2.

That $\mathcal{F}(X)$ is amenable now follows from another application of Theorem 6.2 because $\mathcal{F}(\ell_p \oplus X)$ has a bounded two-sided approximate identity and every compact operator on ℓ_p factors through X . ■

The finite rank projections on \mathcal{L}_p -spaces which were described above almost show that these spaces have property (A). The projections may be used to produce a net of biorthogonal systems satisfying A(i) and A(iii). However, it is not clear that the net will satisfy A(ii). If it could be shown that \mathcal{L}_p -spaces in fact have property (A), then there would be a direct proof of the amenability of $\mathcal{F}(X)$ for these spaces. It seems that indirect arguments are needed to establish many of the properties of \mathcal{L}_p -spaces, see the remark after the statement of Theorem III in [L&R], and so it may be that they do not have property (A). Some specific examples for which this may be tested are the spaces $\ell_2 \oplus \ell_p$. For $1 < p < \infty$, $\ell_2 \oplus \ell_p$ is a \mathcal{L}_p -space, see [L&P], example 8.2, but it is not clear that it has property (A).

Theorem 6.2 may be used to show that $\mathcal{F}(X)$ is amenable for some other spaces which may fail to have property (A). Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers and choose p and q with $1 \leq p, q < \infty$ and $p \neq q$. Put $X = (\bigoplus_{k=1}^\infty \ell_p^{n_k})_{\ell_q}$. Then X has the bounded approximation property and so $\mathcal{K}(X) = \mathcal{F}(X)$. If $\{n_k\}_{k=1}^\infty$ is bounded, then X is isomorphic to ℓ_q and so we will suppose that $\{n_k\}_{k=1}^\infty$ is not bounded. Clearly X is isomorphic to a complemented subspace of $(\bigoplus_{k=1}^\infty \ell_p)_{\ell_q}$ and so every T in $\mathcal{F}(X)$ is a product $T = UV$, where U is in $\mathcal{F}(X, (\bigoplus_{k=1}^\infty \ell_p)_{\ell_q})$ and V in $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p)_{\ell_q}, X)$.

We also have that every T in $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p)_{\ell_q})$ is a product $T = UV$, where U is in $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p)_{\ell_q}, X)$ and V in $\mathcal{F}(X, (\bigoplus_{k=1}^\infty \ell_p)_{\ell_q})$. To see this, for each r let P_r be the natural rank r^2 projection of $(\bigoplus_{k=1}^\infty \ell_p)_{\ell_q}$ onto $(\bigoplus_{k=1}^r \ell_p^r)_{\ell_q}$. Then $\{P_r\}_{r=1}^\infty$ is a bounded left approximate identity for $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p)_{\ell_q})$ and so $T = \sum_{r=1}^\infty P_r T_r$, where $\sum_{r=1}^\infty \|T_r\| < \infty$. Since $\{n_k\}_{k=1}^\infty$ is not bounded, X has a complemented subspace isomorphic to $(\bigoplus_{r=1}^\infty (\bigoplus_{k=1}^r \ell_p^r)_{\ell_q})_{\ell_q}$. Hence we have for each r that $P_r = U_r V_r$, where U_r is in $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p)_{\ell_q}, X)$ and V_r in $\mathcal{F}(X, (\bigoplus_{k=1}^\infty \ell_p)_{\ell_q})$, $\|U_r\| = 1 = \|V_r\|$ and $U_r V_s = 0$ if $r \neq s$. It follows that T factors as required.

Now $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p)_{\ell_q})$ is amenable, see Corollary 4.8. The above remarks about factoring approximable operators therefore allow us to apply Theorem 6.2 to prove

THEOREM 6.5: *Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers. Then the algebra $\mathcal{F}((\bigoplus_{k=1}^\infty \ell_p^{n_k})_{\ell_q})$ is amenable.*

It was remarked above that $\mathcal{F}(\ell_p \oplus \ell_2)$ is amenable. This suggests that $\mathcal{F}(\ell_p \oplus \ell_q)$ may be amenable for all p and q . That this is not so will follow from a further general result about amenable Banach algebras.

Definition 6.6: A Banach algebra \mathcal{B} has trivial virtual centre if, for each b^{**} in \mathcal{B}^{**} with $bb^{**} = b^{**}b$ for all b in \mathcal{B} , there is λ in \mathbb{C} with $bb^{**} = \lambda b = b^{**}b$ for all b in \mathcal{B} .

The algebras in which we are interested have this property.

PROPOSITION 6.7: *Let X be a Banach space. Then $\mathcal{F}(X)$ has trivial virtual centre.*

Proof: Let P be a rank one projection on X . Then $P\mathcal{F}(X)P = \mathbb{C}P$. Suppose that B_0^{**} in $\mathcal{F}(X)^{**}$ satisfies $BB_0^{**} = B_0^{**}B$ for all B in $\mathcal{F}(X)$. Then $PB_0^{**} = P^2B_0^{**} = PB_0^{**}P$. Since the map $B^{**} \mapsto PB^{**}P$ is the second adjoint of the map $B \mapsto PBP$, it follows that there is λ_0 in \mathbb{C} such that $PB_0^{**} = \lambda_0P$. Consequently $\{T \in \mathcal{F}(X) \mid TB_0^{**} = \lambda_0T\}$ is a non-zero closed two-sided ideal in the simple Banach algebra $\mathcal{F}(X)$. Therefore $TB_0^{**} = \lambda_0T$ for all T in $\mathcal{F}(X)$. ■

For the next theorem let \mathcal{A} and \mathcal{A}_{ii} be as above.

THEOREM 6.8: *Suppose that \mathcal{A} is amenable, that \mathcal{A}_{11} and \mathcal{A}_{22} have trivial virtual centre and that \mathcal{A}_{21} and \mathcal{A}_{12} are not both zero. Then $\mathcal{A}_{jj} = \mathcal{A}_{jj}^\circ$ for at least one value of j .*

Proof: Denote $\mathcal{A}^\circ = \{a \in \mathcal{A} \mid P_i a P_i \in \mathcal{A}_{ii}^\circ, i = 1, 2\}$. On \mathcal{A}° define the norm $\|a\|^\circ = \max\{\|P_1 a P_1\|^\circ, \|P_1 a P_2\|, \|P_2 a P_1\|, \|P_2 a P_2\|^\circ\}$. Then, for $a \in \mathcal{A}$, $a^\circ \in \mathcal{A}^\circ$ we have $\|aa^\circ\|^\circ \leq 2\|a\| \|a^\circ\|^\circ$ and $\|a^\circ a\|^\circ \leq 2\|a\| \|a^\circ\|^\circ$. Hence $(\mathcal{A}^\circ, \|\cdot\|^\circ)$ is a Banach \mathcal{A} -bimodule.

The map $a \mapsto P_1 a P_2 - P_2 a P_1 = P_1 a - a P_1$ is a derivation from \mathcal{A} into \mathcal{A}° and so there is C in $(\mathcal{A}^\circ)^{**}$ such that $P_1 a P_2 - P_2 a P_1 = aC - Ca$ for all a in \mathcal{A} . Since $\mathcal{A}^\circ = \bigoplus_{i,j=1,2} P_i \mathcal{A}^\circ P_j$, we have $C = \sum_{i,j=1,2} C_{ij}$, where C_{ij} belongs to $(P_i \mathcal{A}^\circ P_j)^{**}$.

If a_{ii} belongs to \mathcal{A}_{ii} , then $a_{ii}C - Ca_{ii} = 0$. In particular, $a_{ii}C_{ii} - C_{ii}a_{ii} = 0$ for each a_{ii} in \mathcal{A}_{ii} , where C_{ii} belongs to $(\mathcal{A}_{ii}^\circ)^{**}$. Now the second adjoint of the inclusion map $\mathcal{A}_{ii}^\circ \rightarrow \mathcal{A}_{ii}$ embeds $(\mathcal{A}_{ii}^\circ)^{**}$ in $(\mathcal{A}_{ii})^{**}$ and so, since \mathcal{A}_{ii} has trivial

virtual centre for each i , there are λ_1 and λ_2 in \mathbb{C} such that $a_{ii}C_{ii} = \lambda_i a_{ii} = C_{ii}a_{ii}$ for a_{ii} in \mathcal{A}_{ii} , $i = 1, 2$.

Suppose, without loss of generality, that \mathcal{A}_{12} is not zero and choose $a_{12} \neq 0$ in \mathcal{A}_{12} . Since \mathcal{A} is amenable, it has a bounded two-sided approximate identity and so, by Proposition 6.1, \mathcal{A}_{12} is an essential left \mathcal{A}_{11} - and essential right \mathcal{A}_{22} -module. Hence there are a_{11} in \mathcal{A}_{11} , a_{22} in \mathcal{A}_{22} and \tilde{a}_{12} in \mathcal{A}_{12} with $a_{12} = a_{11}\tilde{a}_{12}a_{22}$. We have $a_{12} = a_{12}C - Ca_{12} = a_{12}C_{22} - C_{11}a_{12}$. Substituting for a_{12} we get $a_{12} = a_{11}\tilde{a}_{12}a_{22}C_{22} - C_{11}a_{11}\tilde{a}_{12}a_{22} = \lambda_2 a_{11}\tilde{a}_{12}a_{22} - \lambda_1 a_{11}\tilde{a}_{12}a_{22} = (\lambda_2 - \lambda_1)a_{12}$. Therefore $\lambda_2 - \lambda_1 = 1$ and so at least one of λ_1 and λ_2 is not zero.

Suppose that $\lambda_1 \neq 0$ and put $b^{**} = \lambda_1^{-1}C_{11}$. Then b^{**} belongs to $(\mathcal{A}_{11}^\circ)^{**}$ and $a_{11}b^{**} = a_{11}$ for every a_{11} in \mathcal{A}_{11} . Hence, if $\{b^\alpha\}_{\alpha \in A}$ is a bounded net in \mathcal{A}_{11}° which converges to b^{**} in the weak*- topology, then $\{a_{11}b^\alpha\}_{\alpha \in A}$ converges weakly to a_{11} . It follows, as in [B&D, Proposition 11.4], that there is a net $\{e^\beta\}_{\beta \in B}$, each e^β being a convex combination of b^α 's, which is a right approximate identity for \mathcal{A}_{11} . The approximate identity $\{e^\beta\}_{\beta \in B}$ is bounded, by $\|b^{**}\|$, in $(\mathcal{A}_{11}^\circ, \|\cdot\|^\circ)$ and \mathcal{A}_{11}° is an ideal in \mathcal{A}_{11} . Hence for each a_{11} in \mathcal{A}_{11} and each $\epsilon > 0$ there is $c = a_{11}e^\beta$ with $\|a_{11} - c\| < \epsilon$ and $\|c\|^\circ < 2\|b^{**}\|\|a_{11}\|$. Consequently, for each a_{11} in \mathcal{A}_{11} , there is a series $\sum_i c_i$ in \mathcal{A}_{11}° with $\sum_i \|c_i\|^\circ < \infty$ and $\sum_i c_i = a_{11}$. Therefore $\mathcal{A}_{11}^\circ = \mathcal{A}_{11}$. ■

These last two results may be reformulated to say that the spaces X with $\mathcal{F}(X)$ amenable have a property which is a little like being primary. Recall that a Banach space, X , is **primary** if, for every bounded projection Q on X , either QX or $(I - Q)X$ is isomorphic to X , see [L& T, Definition 3.b.7]. Let us say that X is **approximately primary** if, for every bounded projection Q on X , at least one of the product maps $\pi: \mathcal{F}(X, QX) \hat{\otimes} \mathcal{F}(QX, X) \rightarrow \mathcal{F}(X)$ or $\pi: \mathcal{F}(X, (I - Q)X) \hat{\otimes} \mathcal{F}((I - Q)X, X) \rightarrow \mathcal{F}(X)$ is surjective. Then every primary space is approximately primary as is every space with a subsymmetric basis, see [L&T, Proposition 3.b.8].

Now put $\mathcal{A} = \mathcal{F}(X)$ and suppose that \mathcal{A} is amenable. Let P_1 be the idempotent in $M(\mathcal{A})$ determined by a bounded projection Q on X . Then \mathcal{A}_{11} is isomorphic to $\mathcal{F}(QX)$ and \mathcal{A}_{22} to $\mathcal{F}((I - Q)X)$. Hence, by Proposition 6.7, \mathcal{A}_{ii} has trivial virtual centre for $i = 1, 2$. Clearly, \mathcal{A}_{12} is not zero and so, by Theorem 6.8, $\mathcal{A}_{ii}^\circ = \mathcal{A}_{ii}$ for at least one value of i . It follows that, if $\mathcal{F}(X)$ is amenable, then X is approximately primary.

THEOREM 6.9: *If $1 < p, q < \infty$, $p \neq q$ and neither p nor q is equal to 2, then $\mathcal{F}(\ell_p \oplus \ell_q)$ is not amenable.*

Proof: In view of 6.7, 6.8 and the remarks following it suffices to show that, if $1 < p, q < \infty$, $p \neq q$ and neither is equal to 2, then the product map $\pi: \mathcal{F}(\ell_p, \ell_q) \widehat{\otimes} \mathcal{F}(\ell_q, \ell_p) \rightarrow \mathcal{F}(\ell_p)$ is not surjective.

Suppose that π is surjective. Then, by the open mapping theorem, there is a $K > 0$ such that for each T in $\mathcal{F}(\ell_p)$ we have $T = \pi(\sum_{n=1}^{\infty} U_n \otimes V_n)$, where $\sum_{n=1}^{\infty} \|U_n\| \|V_n\| < K \|T\|$. It follows, since ℓ_q is isomorphic to $(\oplus_{n=1}^{\infty} \ell_q)_{\ell_q}$, that $T = UV$, where U is in $\mathcal{F}(\ell_p, \ell_q)$, V in $\mathcal{F}(\ell_q, \ell_p)$ and $\|U\| \|V\| < K \|T\|$.

Let P_j be the projection onto the span of the first j vectors of the standard basis for ℓ_p . Then, since we are supposing that π is surjective, $P_j = U_j V_j$ where $\|U_j\| \|V_j\| < K$. Put $Q_j = V_j U_j$. Then Q_j is a projection on ℓ_q and $\|Q_j\| < K$. Defining $U'_j = P_j U_j Q_j$ and $V'_j = Q_j V_j P_j$, we have that U'_j is an isomorphism from the range of Q_j to the range of P_j , V'_j is the inverse of U'_j and $\|U'_j\| \|V'_j\| < K^3$. Hence, if π is surjective, then ℓ_p is finitely representable in ℓ_q , see [Wo, Definition II.E.15], in the strong sense that the canonical projections $P_j: \ell_p \rightarrow \ell_p$ factor as described. It is known that this is not so if $p \neq q$ and neither is equal to 2. There are several cases. However, if p' and q' are the conjugate numbers to p and q respectively, we obtain a similar factorization of $P_j: \ell_{p'} \rightarrow \ell_{p'}$ through $\ell_{q'}$ just by taking conjugate maps. Hence we need only consider the following three cases.

First, suppose that $p < 2 < q$. If ℓ_p were finitely representable in ℓ_q , then, since ℓ_q is of type 2, ℓ_p would be of type 2. (See [Wo, Definition III.A.17 and Theorem III.A.23]) That is not so. Therefore ℓ_p is not finitely representable in ℓ_q .

Next, suppose that $2 < q < p$. If ℓ_p were finitely representable in ℓ_q , then, since ℓ_q is of cotype q , ℓ_p would be of cotype q . Since ℓ_p is not of cotype q , ℓ_p is not finitely representable in ℓ_q .

Finally, suppose that $2 < p < q$. If ℓ_p were finitely representable in ℓ_q , then ℓ_p would be isomorphic to a subspace of $L_q(\mu)$ for some measure μ , see Proposition 7.1 in [L&P]. It would then follow, by Corollary 2 in [K&P], that ℓ_p had a complemented subspace isomorphic to ℓ_2 or ℓ_q . However, that is not possible because, by Proposition 2.c.3 in [L&T], every operator from ℓ_p to ℓ_2 and every operator from ℓ_q to ℓ_p is compact. This argument is also sketched on [Wo] pages 104 and 107. ■

The above proof also shows that $\mathcal{F}(c_0 \oplus \ell_p)$ is not amenable when $p < 2$ but does not treat the case $p > 2$. Similarly, $\mathcal{F}(\ell_1 \oplus \ell_p)$ is not amenable when $p > 2$.

We conclude this section with a result which shows that amenability of $\mathcal{F}(X)$ is partially preserved on complemented subspaces of X .

6.10 THEOREM: *Let X and Y be Banach spaces and suppose that $\mathcal{F}(X \oplus Y)$ is amenable. Then at least one of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ is amenable.*

Proof: Put $\mathcal{A} = \mathcal{F}(X \oplus Y)$ and let P_1 be the idempotent in $M(\mathcal{A})$ determined by the projection onto X with kernel Y . Then, by 6.7 and 6.8, $\mathcal{A}_{jj} = \mathcal{A}_{jj}^o$ for at least one value of j . By Theorem 6.2, it follows that at least one of \mathcal{A}_{11} and \mathcal{A}_{22} is amenable. Since \mathcal{A}_{11} is isomorphic to $\mathcal{F}(X)$ and \mathcal{A}_{22} is isomorphic to $\mathcal{F}(Y)$, the result follows. ■

The conclusion of this last theorem is the best possible, that is, there are spaces X and Y such that $\mathcal{F}(X \oplus Y)$ is amenable but $\mathcal{F}(X)$ is not. For example, let $X = c_0 \oplus \ell_1$ and $Y = \ell_1(c_0) = \left(\bigoplus_{n=1}^{\infty} c_0\right)_{\ell_1}$. Then $X \oplus Y$ is isomorphic to Y . Hence $\mathcal{F}(X \oplus Y)$ and $\mathcal{F}(Y)$ are amenable by Corollary 4.8. On the other hand, $\mathcal{F}(X)$ is not amenable by Theorem 6.9.

7. Open questions and conclusion

The name ‘amenable’ is used for a Banach algebra \mathcal{A} satisfying the cohomological condition $H^1(\mathcal{A}, X^*) = 0$ for all Banach \mathcal{A} -modules X , see [J], because of the theorem that a group algebra $L^1(G)$ satisfies this condition if and only if the locally compact group G is amenable, [J]. Amenability is an important property of groups which has many characterizations. As well as the cohomological characterization of the group algebra, it may be described in terms of group representations, fixed points of group actions, translation invariant functionals and in other ways. The Følner conditions on compact subsets of the group characterize amenability in terms of properties intrinsic to the group. Alternative characterizations of the amenability of $\mathcal{K}(X)$ and $\mathcal{F}(X)$ would help us to have a better understanding of its significance. We are thus led to ask

QUESTION 7.1: *What are the intrinsic properties of the Banach space X which are equivalent to amenability of $\mathcal{K}(X)$ and $\mathcal{F}(X)$?*

The results we have obtained so far suggest that amenability of $\mathcal{K}(X)$ and

$\mathcal{F}(X)$ may be equivalent to some sort of approximation property for X . Such an approximation property, if it were to exist, would be the analogue of the Følner conditions.

Approximation properties are certainly necessary. Since an amenable algebra has a bounded two-sided approximate identity, if $\mathcal{K}(X)$ is amenable, then X^* has what is called in [G&W] the $B - \mathcal{K}(X)^a$ - AP, and in [Sa] the $*$ - b.c.a.p., that is, the identity operator on X^* is approximable in the topology of convergence on compacta by operators which are adjoints of compact operators on X . It also follows, by [D, Theorem 2.6], that X has the bounded compact approximation property. Similarly, if $\mathcal{F}(X)$ is amenable, then X and X^* have the bounded approximation property. However, the relationship between amenability of $\mathcal{K}(X)$ and the approximation property is not clear.

QUESTION 7.2: *Does amenability of $\mathcal{K}(X)$ imply that X has the approximation property?*

If there should be a Banach space X which does not have the approximation property but is such that $\mathcal{K}(X)$ is amenable, then $\mathcal{K}(X)/\mathcal{F}(X)$ would be a radical, amenable Banach algebra. At present no example of such a Banach algebra is known.

Theorem 6.9 shows that for $\mathcal{F}(X)$ to be amenable it does not suffice that X^* have the $B - \mathcal{F}(X)^a$ - AP. It seems necessary for there also to be some sort of symmetrization of the approximation property. We have seen, in Section 4, a symmetrized approximation property, property (A), which forces the amenability of $\mathcal{F}(X)$. This property was used to show that $\mathcal{F}(X)$ is amenable for many of the classical Banach spaces and for spaces with a shrinking, subsymmetric basis.

QUESTION 7.3: *Is property (A) or some similar symmetrized approximation property equivalent to amenability of $\mathcal{F}(X)$ or $\mathcal{K}(X)$?*

In order to determine how close this property is to being equivalent to the amenability of $\mathcal{F}(X)$, it would be useful to investigate whether $\mathcal{F}(X)$ is amenable if X is a space which is clearly unlikely to have this symmetrized approximation property. Examples that come to mind are the James space, which does not have an unconditional basis ([L& T, 1.d.2]), and the Tsirelson space, which contains no subsymmetric basic sequence ([L& T, p. 132]).

QUESTION 7.4: *Is $\mathcal{F}(X)$ amenable if X is the James space or the Tsirelson space?*

We have seen that the class of spaces, X , such that $\mathcal{F}(X)$ is amenable is not closed under direct sums or under passing to complemented subspaces. However, any space, X , such that $\mathcal{F}(X)$ is amenable has the property that X^* satisfies the $B - \mathcal{F}(X)^a$ -AP and this property is inherited by complemented subspaces of X and is preserved under direct sums. Perhaps this is the most that can be said about such spaces.

QUESTION 7.5: *Is the smallest space ideal containing all spaces, X , such that $\mathcal{F}(X)$ is amenable equal to the class of all Banach spaces, X , such that X^* has the $B - \mathcal{F}(X)^a$ -AP?*

Recall from [Pie, Definition 2.1.1], that a **space ideal** is a class of Banach spaces which contains the finite dimensional spaces and is closed under direct sums and taking complemented subspaces. It is clear that the class of spaces such that X^* has the $B - \mathcal{F}(X)^a$ -AP is a space ideal. Should the answer to 7.2 be 'no', an obvious further question would be whether the class of all Banach spaces whose duals have the $B - \mathcal{K}(X)^a$ -AP is equal to the smallest space ideal containing all spaces, X , such that $\mathcal{K}(X)$ is amenable.

It was shown by J. Lindenstrauss, see [L&T, Theorem 3.b.1], that every Banach space with an unconditional basis is isomorphic to a complemented subspace of a space with a symmetric basis. In view of the possible equivalence of the amenability of $\mathcal{F}(X)$ with some symmetric approximation property, this suggests the following refinement of 7.5.

QUESTION 7.6: *Is every Banach space, X , such that X^* has the $B - \mathcal{F}(X)^a$ -AP isomorphic to a complemented subspace of a space, Y , such that $\mathcal{F}(Y)$ is amenable?*

The spaces C_p , $1 \leq p < \infty$, introduced by W. B. Johnson [Jo1] will provide the answer to this last question. The space C_p is the ℓ^p direct sum of a sequence of finite dimensional spaces which is dense in the set of all finite dimensional spaces. It has the property that every approximable operator factors through it and, for $1 < p < \infty$, $\mathcal{F}(C_p)$ has a bounded two-sided approximate identity. Now let X be any space such that X^* has the $B - \mathcal{F}(X)^a$ -AP. Then, since C_p has the above properties, Theorem 5.2 implies that $\mathcal{F}(X \oplus C_p)$ is amenable if and only if $\mathcal{F}(C_p)$ is amenable. Therefore the answer to 7.6 is 'yes' if $\mathcal{F}(C_p)$ is amenable. On the other hand, if C_p is isomorphic to a complemented subspace of some space,

Y , such that $\mathcal{F}(Y)$ is amenable, that is, if the answer to 7.6 is 'yes' when $X = C_p$, then $\mathcal{F}(C_p)$ is amenable.

QUESTION 7.7: *Is $\mathcal{F}(C_p)$ amenable for any, and hence all, $1 < p < \infty$?*

Note that C_1^* does not have the approximation property, see [Jo2, Theorem 3], and so $\mathcal{F}(C_1)$ is not amenable.

Another theorem, similar to that of Lindenstrauss, is proved in [J, R&Z] and [P], see [L&T, Theorem 1.e.13]. It says that any separable Banach space with the B.A.P. is isomorphic to a complemented subspace of a Banach space with a basis. There is an even stronger theorem, see [L&T, Theorems 2.d.8 and 2.d.10], that there is a Banach space, U , with basis such that any separable Banach space with the B.A.P. is isomorphic to a complemented subspace of U and that U is determined uniquely up to isomorphism by this property. The space U is said to be complementably universal for the spaces with the B.A.P. Now if X has a shrinking basis, then X^* has the $B - \mathcal{F}(X)^a$ -AP. This suggests

QUESTION 7.8:

- (a) *Is there a Banach space, V , with a shrinking basis which is complementably universal for the spaces, X , such that X^* has the $B - \mathcal{F}(X)^a$ -AP?*
- (b) *If so, is $\mathcal{F}(V)$ amenable?*

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